Schur's Inequality — And Its Mystery

Issai Schur (1875-1941) was a mathematical magician who discovered many amazing tricks. I don't know what led him to his namesake inequality, but to me it is the most mysterious of the completely elementary inequalities. We'll ponder its mystery after a quick look at the inequality and its simple proof.

SCHUR'S INEQUALITY: For nonnegative x, y, z and for $r > 0$ one has the symmetric inequality

$$0 \le x^r(x-y)(x-z) + y^r(y-x)(y-z) + z^r(z-x)(z-y). \qquad (1)$$

Moreover, we have equality here if and only if either (a) all three variables are equal or (b) a pair of the variables are equal and the third is zero.

The symmetry is elegant, but it does not suggest any way to proceed, so let's break the symmetry and suppose (without loss of generality) that $0 \le x \le y \le z$. At a minimum, this lets us identify some positive terms. For example, we now see that the first summand is nonnegative, so we can look at what's left. The common factor is of these two terms is $z - y$, and when we take it out our sum becomes

$$x^r(x-y)(x-z) + (z-y)\{z^r(z-x) - y^r(y-x)\}. \qquad (2)$$

By grace of the Fates, we now see why the second term is also nonnegative — both factors of $z^r(z-x)$ are at least as large as the factors of $y^r(y-x)$. Finally, we can read off the case of equality from the representation (2) using the observation that for the sum to be zero, both summands must be zero.

Mystery — To Minimize or Maximize?

Part of the strangeness of (1) comes from the rth powers, but the proof shows that these powers are a red herring. We can replace them by any nonnegative nondecreasing function of the variables. This suggests that Schur's inequality is perhaps best viewed as a *consequence of order* (CSMC, Chapter 5).

Still, the *mystery of three* remains — at least for me. Can you find a four variable analog to Schur's inequality? Let me know if you do.

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KEYWORDS: Schur's inequality, symmetric inequality, mathematical inequalities, mathematical olympiad, Issai Schur, consequences of order.